By way of example we have depicted in Fig. 1 the profile of the bottom of the crater in a copper target for $q_0 = 10^6$ W/cm^2 and $q_0/q(b) = 2$. It can be seen from Fig. 2 that the velocity of motion of the front g depends significantly on its curvature. Even when the energy density is uniformly distributed over the cross section of the beam, i.e., when $q_0 = q(b)$ (curve 2), the velocity g is appreciably less than the velocity of a planar front (curve 1). While if $q_0/q(b) = 2$ (curves 3 and 4), neglecting the curvature of the front leads to almost a 10-fold error in the determination of g in the range 10^6–10^7 W/cm^2.

Figure 3 shows the dependence of the rate of motion of the front g and the temperature T_1 on the parameter v_0. As pointed out in [1], formula (10') very roughly determines the pre-exponential factor in the kinetic equation. The calculations show, however, that the magnitude of this parameter has only a small effect on the results of the computations; varying v_0 by an order (from 10^5 to 10^6 cm/sec) changes the velocity g only by around 5%.

NOTATION

T_1, temperature of the center of the bottom of the crater; T_2, temperature of the outermost points of the front (at r = b); g, stationary rate of deepening of the crater; $p = (d^2T/d\xi^2)/(dT/d\xi)$; $\xi_2 = (1/B^2) \times \ln(T_1/T_2)$, distance along the X axis from the central point of the front to the isotherm with temperature T_2; $h = (1/B^2) \ln (T_1/T_m)$, depth of the molten layer along the X axis; $r_1 = (2,4/AB)$.

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GROUP PROPERTIES OF THE NONLINEAR HEAT-CONDUCTION EQUATION AND THE SOLUTION OF INVERSE PROBLEMS

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We develop a numerical—experimental method of determining the thermophysical properties of materials in which we use group-invariant solutions of the nonlinear heat-conduction equation. We study the stability of a class of such solutions.

In [1] Ovsyannikov examined the problem of the group classification of the equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x} \left(f(u) \frac{\partial u}{\partial x} \right), \tag{1}$$

i.e., the problem of determining the fundamental group admitted by Eq. (1) for various forms of the function f of the unknown solution.

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Equation (1) is obtained from the more general equation

$$F_1 \equiv \rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) = 0 \tag{2}$$

through introduction of the new function u and the coefficient f:

$$u = \int \rho c dT, \qquad f(u) = k/\rho c. \tag{3}$$

In the same sense as the group classification problem solved in [1] this device appears to be a natural one, as does also the repeated neglect of arbitrary constants (all considerations in [1] are taken "to within equivalency," i.e., linear transformations of arbitrary variables are ignored). The problems we solve in this paper, on the other hand, require retaining (in explicit form) a maximum number of the free parameters and functions, since our goal is to use the invariant solutions of the heat-conduction equation to solve inverse problems, i.e., to determine the coefficients of heat capacity and heat conduction, namely, $c\rho$ and k, as functions of the temperature.

1. Invariant Solutions of Eq. (2)

Equation (2) can be conveniently written in the equivalent form of a system of first-order quasilinear equations; thus,

$$F_1 \equiv p_1^1 - \psi p_2^2 = 0; \qquad F_2 \equiv \varphi p_2^1 - u^2 = 0. \tag{4}$$

Here we have introduced the notation

$$t = x^1, \quad x = x^2, \quad u = u^1, \quad \varphi \frac{\partial u}{\partial x} = u^2, \quad p_j^i = \frac{\partial u^i}{\partial x^j}.$$

The conditions for invariance of the differential manifold (4) with respect to the operator X are of the form

$$\tilde{X}F_1 = 0, \quad \tilde{X}F_2 = 0, \quad F_1 = 0, \quad F_2 = 0, \tag{5}$$

where

$$X = \xi_x^i \frac{\partial}{\partial x^i} + \xi_u^k \frac{\partial}{\partial u^k}; \quad \tilde{X} = X + \xi_{pi}^k \frac{\partial}{\partial p_i^k};$$

$$\xi_{pi}^k = D_i(\xi_u^k) - p_j^k D_i(\xi_x^j); \quad D_i \equiv \frac{\partial}{\partial x^i} - p_i^l \frac{\partial}{\partial u^l}$$

(summation is taken over repeated Latin indices). The defining system of equations is represented by the following relationships [expanded form of the conditions (5)]:

$$\begin{gathered} \xi_{p1}^1 - p_2^2 \dot{\psi} \xi_u^1 - \psi \xi_{p2}^2 = 0, \\ p_2^1 \dot{\varphi} \xi_u^1 + \varphi \xi_{p2}^1 - \xi_u^2 = 0, \\ p_1^1 = \psi p_2^2, \quad \varphi p_2^1 = u^2. \end{gathered} \tag{6}$$

The dot above the functions φ and ψ denotes differentiation with respect to u. Relative to the coordinates ξ_x^i and ξ_u^k of the operator X, the system (6) is decomposable (owing to the arbitrariness and independence of the variables p_1^2, p_2^2); its general solution can be written in the form

$$\xi_t = \alpha t + \beta; \quad \xi_x = -\frac{\gamma}{4} x^2 + \delta x + \varepsilon;$$

$$\xi_u = \frac{\gamma x + \zeta}{\dot{\chi}}; \quad \xi_q = (\omega \gamma x + \varkappa) q + \frac{\gamma \varphi}{\dot{\chi}}.$$

Here

$$q = \varphi \frac{\partial u}{\partial x}, \quad \chi = \ln(\varphi \psi).$$

The arbitrary constants α, β, γ, δ, ε, ζ, $\varkappa$, ω are determined from conditions introduced below. It is important to distinguish the following cases.

a) $\chi(u)$ an arbitrary function. In this case α, β, ε can take on arbitrary values, and $\delta = \alpha/2$, $\varkappa = -\delta$, $\gamma = \zeta = 0$. The invariant solutions corresponding to this case cannot be used to solve inverse problems without some additional assumptions on the form of the functions $\varphi(u)$ and $\psi(u)$ (see [2]).

b) The function $\chi(u)$ satisfies the equation

$$\ddot{\chi} - \dot{\chi}\frac{\dot{\varphi}}{\varphi} = \omega(\dot{\chi})^2 \tag{7}$$

or (what amounts to the same thing) the equation

$$\ddot{\chi} + \dot{\chi}\dot{\psi}/\psi = (1+\omega)(\dot{\chi})^2. \tag{8}$$

A constraint of this kind is introduced in such a way that if the function $\varphi(\psi)$ is chosen arbitrarily, then the other function ψ (respectively, φ) is obtained by solving Eq. (7) [respectively, Eq. (8)]. In this case the constants α, β, ε, ζ can be selected arbitrarily; γ, δ, $\varkappa$ and ω are chosen by means of the conditions

$$\gamma(1+4\omega) = 0, \qquad \alpha + \zeta - 2\delta = 0, \qquad \varkappa + \delta + \omega\zeta = 0.$$

It is convenient to have an explicit form of the relationship between φ and ψ, specified by the conditions (7) and (8). Integrating Eq. (8), we find

$$\varphi = c(u)\left(C_1 + C_2\int_0^u c(v)\,dv\right)^{-\frac{1}{1+\omega}}, \qquad c \equiv 1/\psi. \tag{9}$$

Similarly, starting from Eq. (7), we readily obtain

$$c(u) = \varphi(u)\exp\left[C_1\int_0^u \varphi(v)\exp\left(\omega\int_0^v \frac{ds}{\varphi(s)}\right)dv + C_2\right]. \tag{10}$$

Here C_1 and C_2 are arbitrary constants of integration.

We consider the cases $\gamma = 0$ and $\gamma \neq 0$ separately.

b.1) $\gamma = 0$. We obtain a set of linearly independent operators of a Lie algebra from the general form of an operator of the one-parameter subgroup:

$$X(\alpha, \beta, \ldots, \omega) = (\alpha t + \beta)\frac{\partial}{\partial t} + \left(\frac{\gamma}{4}x^2 + \delta x + \varepsilon\right)\frac{\partial}{\partial x} + \frac{\gamma x + \zeta}{\dot{\chi}}\frac{\partial}{\partial u} + \left[(\omega\gamma x + \varkappa)q + \frac{\gamma\varphi}{\dot{\chi}}\right]\frac{\partial}{\partial q},$$

taking, in turn, one of the four free parameters α, β, ε, ζ to be nonzero and determining the remaining parameters from the relations $\alpha + \zeta = 2\delta$, $\varkappa + \beta + \omega\zeta = 0$. As a basis we can thus choose the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - q\frac{\partial}{\partial q},$$

$$X_4 = x\frac{\partial}{\partial x} + \frac{2}{\dot{\chi}}\frac{\partial}{\partial u} - (1+2\omega)q\frac{\partial}{\partial q}.$$

We consider the possible forms of the invariant solutions in this case.

b.1.1) $X = X_1$; the invariants are $J_1 = x$, $J_2 = u$; $u = u(x)$.

b.1.2) $X = X_2$; the invariants are $J_1 = t$, $J_2 = u$; $u = u(t)$.

b.1.3) $X = X_3$; the invariants are $J_1 = x/\sqrt{t}$, $J_2 = u$, $J_3 = q\sqrt{t}$.

The solution has the form

$$u = f(J_1); \quad q = F(J_1)/\sqrt{t}.$$

b.1.4) $X = X_1 - \alpha X_2$; the invariants are $J_1 = \alpha t + x$, $J_2 = u$, $J_3 = q$.

The solution has the form

$$u = f(J_1); \quad q = F(J_1).$$

b.1.5) $X = X_4$; the invariants are $J_1 = t$, $J_2 = \chi(u) - 2\ln x$, $J_3 = qx^{(1+2\omega)}$.

The solution is given by the relationships

$$\chi[u(x, t)] = f(t) + 2\ln x; \qquad q = x^{-(1+2\omega)}F(t).$$

b.1.6) $X = X_4 - \alpha X_1$; the invariants are $J_1 = e^t x^{\alpha}$, $J_2 = \chi(u) - 2\ln x$, $J_3 = qx^{(1+2\omega)}$.

The solution is determined from the relationships

$$\chi[u(x,\ t)] = f(J_1) + 2\ln x; \qquad q = x^{-(1+2\omega)}F(J_1).$$

b.1.7) $X = (1/2)(X_4 - X_3)$; the invariants are $J_1 = x$, $J_2 = \chi + \ln t$, $J_3 = q^{t-\omega}$.

The solution is determined by means of the relationships

$$\chi[u(x,\ t)] = f(J_1) - \ln t; \qquad q = t^{\omega}F(J_1).$$

b.1.8) $X = X_3 + (1/2)(X_3 - X_4)$; the invariants are $J_1 = x/\sqrt[3]{t}$, $J_2 = \dot{\chi} + (1/3)\ln t$, $J_3 = qx^{(1-\omega)}$.

The solution is determined by means of the relationships

$$\chi[u(x,\ t)] = f(J_1) - \frac{1}{3}\ln t; \quad q = x^{(\omega-1)}F(J_1).$$

b.1.9) $X = (1/2)(X_4 - X_3) + \alpha X_2$; the invariants are $J_1 = e^x t^{\alpha}$, $J_2 = \chi(u) - (1/\alpha)x$, $J_3 = qt^{-\omega}$.

The solution is determined by means of the relationships

$$\chi[u(x,\ t)] = f(J_1) + \frac{1}{\alpha}x; \qquad q = t^{\omega}F(J_1).$$

The functions $f(J_1)$ and $F(J_1)$, which appear in the expressions for $u(x,\ t)$ and $q(x,\ t)$, are obtained by solving ordinary differential equations; these latter are obtained, in turn, by substituting u and q into the system of equations (4). As an example, we consider the case

$$\chi(u) = f(x) - \ln t, \qquad q = t^{\omega}F(x).$$

Using the relations

$$\dot{\chi}\frac{\partial u}{\partial t} = -\frac{1}{t}, \quad \dot{\chi}\frac{\partial u}{\partial x} = \frac{df}{dx}, \quad \ddot{\chi}\left(\frac{\partial u}{\partial x}\right)^2 + \dot{\chi}\frac{\partial^2 u}{\partial x^2} = \frac{d^2 f}{dx^2}$$

and Eq. (4), we readily see that f(x) is a solution of the equation

$$f'' - \omega(f')^2 + e^{-f} = 0. \tag{11}$$

Correspondingly, for F(x) we obtain the equation

$$\frac{d\ln F}{dx}\cdot\frac{de^f}{dx} + 1 = 0. \tag{12}$$

The solution of Eqs. (11) and (12) may be reduced to quadratures:

$$\pm\int\limits_{\exp C_4}^{y}\frac{d\xi}{\sqrt{\frac{2\xi}{1+2\omega}\left(1+C_3\xi^{(1+2\omega)}\right)}} = x - x_0, \quad 1+2\omega \neq 0; \tag{13}$$

$$x - x_0 = \int\limits_{C_4}^{f}\frac{\exp(\xi/2)}{(C_3-\xi)}\,d\xi, \quad 1+2\omega = 0; \tag{14}$$

$$F(x) = C_4\exp\int\limits_{x}^{x_0}\left(\frac{dy}{d\xi}\right)^{-1}d\xi; \quad y(\xi) = \exp f(\xi). \tag{15}$$

Here C_3, C_4, C_5 are constants of integration.

We return now to the case $\gamma \neq 0$, i.e., $\omega = -1/4$.

b.2) $\gamma \neq 0$. In addition to the operators X_1, X_2, X_3, and X_4, we can write down yet another linearly independent operator upon varying γ:

$$X_5 = \frac{x^2}{4}\frac{\partial}{\partial x} + \frac{x}{\dot{\chi}}\frac{\partial}{\partial u} + \left(\frac{\varphi}{\dot{\chi}} + \omega xq\right)\frac{\partial}{\partial q};$$

we can then also write down new types of invariant solutions connected with this operator.

b.2.1) $X = X_5$; the invariants are $J_1 = t$, $J_2 = \chi - 4 \ln x$.

The solution is given by the relationship

$$\chi(u) = f(t) + 4 \ln x.$$

b.2.2) $X = X_5 - \alpha X_1$; the invariants are $J_1 = t - 4\alpha/x$, $J_2 = \chi - 4 \ln x$.

The solution is given by the relationship

$$\chi(u) = f(J_1) + 4 \ln x.$$

b.2.3) $X = X_5 + \alpha(X_4 - X_3)$; the invariants are $J_1 = e^{1/x} t^{1/8\alpha}$, $J_2 = \chi(u) + (8\alpha/x) - 4 \ln x$.

The solution is given by the relationship

$$\chi(u) = f(J_1) + 4 \ln x - \frac{8\alpha}{x}.$$

By forming linear combinations of the operators $X_1, X_2, \ldots, X_5$ we can obtain many special forms of invariant solutions of the system (4). It is a well-known fact that there are only a few essentially distinct solutions, i.e., solutions not obtainable from one another by applying a particular transformation of the fundamental group. From the point of view of applications, however, it is of no consequence whether the applicable solutions are or are not essentially distinct; the only thing of importance is that the solutions be of a "suitable" form.

2. Application to the Solution of Inverse Problems

Consider the following inverse problem from the theory of heat conduction. We wish to find, for a given solution of the boundary-value problem,

$$\frac{\partial u}{\partial t} - \psi \frac{\partial}{\partial x}\left(\varphi \frac{\partial u}{\partial x}\right) = 0, \quad x \in (x_0, x_1), \quad t \in [t_0, t_1],$$
$$u(x_0, t) = \varphi_0(t), \quad u(x_1, t) = \varphi_1(t), \quad u(x, t_0) = u^0(x), \qquad (16)$$

where φ and ψ are positive functions of the temperature belonging to the classes C^0 and C^1, a pair of functions (φ, ψ) which, when substituted into Eq. (16), would make the latter an identity. In constructing a solution of this problem we can use the invariant solutions introduced in Sec. 1. We give an example below illustrating this approach in detail.

We consider an invariant solution corresponding to the operator $X_4 - X_3$. The temperature field $u(x, t)$, in this case, must satisfy the condition

$$\chi[u(x, t)] = f(x) - \ln t,$$

the heat flow $q = t^{\omega} F(x)$. The assumption we make as to the possible form of the functions $\varphi(u)$ and $\psi(u)$ is conditioned by the fact that these functions are connected through one of the pair of relations (9) and (10). The functions $f(x)$ and $F(x)$ are given by the expressions (13)-(15). It is important to note that these latter functions do not contain any of the functions associated with the unknown characteristics $c(u)$ and $\varphi(u)$; they can be determined numerically or can be obtained in the form of approximate analytic expressions. If $f^*(x, C_3, C_4)$ is the solution of Eq. (11), the temperature field $u(x, t)$ can now be found from the relationship

$$C_1 + C_2 \int_0^{u(x,t)} c(v)\, dv = \exp[(1 + \omega)(\ln t - f^*(x, C_3, C_4))].$$

It is convenient to introduce the new time variable $\tau = t^{\omega}$. In terms of this new time variable the heat flux can be written in the form

$$q(\tau) = F^*(x, C_3, C_4, C_5)\tau.$$

Let the following conditions be satisfied:

a) the unknown heat capacity is an element of an m-parameter family of functions; thus,

$$c = c(u, b_1, b_2, \ldots, b_m);$$

b) at the ends of the interval $[x_0, x_1]$ boundary conditions of the form

$$q(x_p, \tau) = A_p \tau; \quad p = 0, 1; \quad \tau \geqslant \tau_0,$$

are maintained, where the constants A_p are arbitrary, but are such that for some values of the constants of integration C_3, C_4, and C_5 the following equations are satisfied:

$$F^*(x_p, C_3, C_4, C_5) = A_p; \qquad p = 0, 1;$$

c) the values of the temperature $u_{pj} \equiv u(x_p, \tau_j)$ are known at the points x_p at the times τ_j (j = 1, 2, ...) [the general number of values of u_{pj} is (m + 5)];

d) the initial distribution $u(x_0, \tau)$ satisfies the conditions of invariance.

Then the solution $\{c(u, \vec{b}); \varphi(c, C_1, C_2, \omega)\}$ of the inverse problem in question is determined from the system of equations

$$C_1 + C_2 \int_0^{u_{pj}} c(v, b_1, b_2, \ldots, b_m)\, dv = \exp\left[(1+\omega)\left(\frac{1}{\omega} \ln \tau_j - f^*(x_p, C_3, C_4)\right)\right],$$

$$p = 0, 1; \qquad j = 1, 2, \ldots$$

Indeed, upon eliminating the unknown quantities $f^*(x_p, C_3, C_4)$, we obtain a system of (m + 3) equations involving the constants C_1 and C_2 and the parameters ω, $b_1, \ldots, b_m$.

The assumptions made above are completely realistic and attainable in practice, except for the one pertaining to the possibility of realizing an invariant distribution at the initial time τ_0. This latter condition, however, can be dropped. The fact of the matter is that the invariant solutions possess a peculiar stability, which amounts to the following: if at the boundary points x_p (p = 0, 1) the heat flux $q(x_p, \tau)$ or the temperature $u(x_p, \tau)$ satisfies the conditions of invariance, then, independently of the initial distribution $u(x, \tau_0)$, the solution $u(x, \tau)$ converges asymptotically to the invariant solution. This can be seen as follows. Consider the function $\Phi(x, t)$, defined by the expression

$$\Phi(x, t) = \chi[u(x, t)] + \ln t.$$

The required property is obviously equivalent to the following condition:

$$\lim_{t \to \infty} \Phi(x, t) = f(x).$$

For the function $F = \exp(-\omega\Phi)$ the boundary conditions are stationary and have the most convenient form

$$\frac{\partial F}{\partial x}(x_p, t) = B_p, \quad p = 0, 1.$$

The equation for $F(x, t)$, subject to perturbations of the initial conditions ($\Phi(x, t_0) \neq f$), is obtained by substituting $F[\Phi(u)]$ into Eq. (16):

$$\frac{\partial F}{\partial s} = K[F] \equiv -\omega F + F^{-\frac{1}{\omega}} \frac{\partial^2 F}{\partial s^2}; \quad s = \ln t; \quad F(x, t_0) = \exp[-\omega\Phi(x, t_0)]. \tag{17}$$

It is readily seen that the function $F_*(x) = \exp(-\omega f)$ is a stationary solution of Eq. (17). We show now that this solution is uniformly and asymptotically stable (see [3]).

The nonlinear differential operator K on the right side of Eq. (17) has at the point F_* the derivative K'_{F_*}:

$$K'_{F_*}[y] = -(1+\omega)y + e^f \frac{\partial^2 y}{\partial x^2}.$$

Considering the function $w = \exp(-f)$ as an operator (uniformly positive), we can state that the derivative K'_{F_*} is a w uniformly dissipative operator. In fact,

$$(wK'_{F_*}[y], y) = -(1+\omega)\|y\|_w^2 - \int_{x_0}^{x_1} \left(\frac{\partial y}{\partial x}\right)^2 dx.$$

Thus, by virtue of the generalized theorem of Lyapunov (see [3]), the spectrum of the operator K'_{F_*} lies in the left half-plane, and, consequently (see [3]), the stationary solution $F_* = \exp(-\omega f(x))$ is uniformly and asymptotically stable. Fairly accurate estimates of the rate of decrease of the norm of the perturbation $h = F - F_*$ can be obtained from the expression

$$\frac{d}{ds}\|h\|^2 = -\int^{x_1} h^2\left[3 + 2\omega \mp \frac{2(1+\omega)}{1+2\omega}\left(1+C_3 y^{(1+2\omega)}\right)\right] dx - 2\int^{x_1} y\left(\frac{\partial h}{\partial y}\right)^2 dx.$$

Here

$$y = \exp\left[f^*(x, C_3, C_4, \omega)\right].$$

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